Constructions of Turán systems that are tight up to a multiplicative constant

Oleg Pikhurko
University of Warwick

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Turán systems

- $G \subseteq \binom{V}{r}$ is a Turán (s,r)-system:
$$\forall X \in \binom{V}{s}\ \exists Y \in G \quad Y \subseteq Z$$
  - r-graph that covers all s-sets
- $T(n,s,r) := \min\left\{|G| : \text{Turán } (s,r)\text{-system } G \subseteq \binom{[n]}{r}\right\}$
- $T(n,s,r) = \binom{n}{r} - \mathrm{ex}(n, K_s^r)$
- Density $t(s,r) := \lim_{n\to\infty} \frac{T(n,s,r)}{\binom{n}{r}} = 1 - \pi(K_s^r)$

$r \leqslant 2$

- $T(n, s, 1) = n - s + 1$
  - $t(s, 1) = 1$
- Mantel'1907: $T(n, 3, 2) = \binom{\lfloor n/2 \rfloor}{2} + \binom{\lceil n/2 \rceil}{2}$
  - $t(3, 2) = \frac{1}{2}$
- Turán'41: $T(n, s, 2)$ is attained by $s - 1$ cliques
  - $t(s, 2) = \frac{1}{s-1}$

The Tetrahedron Problem

- $t(4,3) \leqslant \frac{4}{9}$
- Turán: Is $T(n,4,3)$ attained by the 3-part construction ?
- Katona-Nemetz-Simonovits'64, de Caen'88, Chung-Lu'99, Razborov'10: $t(4,3) \geqslant 0.43833\ldots$
- Brown'83: other constructions
- Kostochka'82, Fon-der-Flaas'88, Frohmader'08: $\Rightarrow$ exponentially many non-isomorphic extremal 3-graphs
- Fon-der-Flaas'88: $G(\text{digraph } D) := \{X \in \binom{V}{3} : D[X]$ has a vertex of degree 0 or out-degree $2\}$. If D has no induced directed 4-cycle then G is a $(4,3)$-Turán system.
- Razborov'11: $|G(D)| \geqslant (\frac{4}{9} + o(1))\binom{n}{3}$ if $\overline{D}$ is complete multipartite or has density $\geqslant \frac{2}{3} - \varepsilon$, some constant $\varepsilon > 0$

Turán $(s, 3)$-systems

- $t(s,3) \leqslant \frac{4}{(s-1)^2}$
- $T(n,5,3) \leqslant \binom{\lfloor n/2 \rfloor}{3} + \binom{\lceil n/2 \rceil}{3}$
- Conjecture (Ringel'64, Turán'70): this is equality
- Surányi'71, Kostochka, Sidorenko'83: false for odd $n \geqslant 9$
- Conjecture: $t(s,3) = \frac{4}{(s-1)^2}$
- Razborov'10:
  - $t(5,3) \geqslant 0.230...$ $(\leqslant 0.25)$
  - $t(6,3) \geqslant 0.141...$ $(\leqslant 0.16)$
- Giraud'90, Markström'09: $\frac{5}{16} = 0.325 \geqslant t(5,4) \geqslant 0.263...$
- Erdős: \$500 for determining $t(s,r)$, some $s > t \geqslant 3$
- Sidorenko'95: No *"plausible conjecture"* in other cases

Lower bounds for $r \geqslant 4$

- Double counting: $t(r+1,r) \geqslant \frac{1}{r+1}$
- Sidorenko'82, de Caen'83, Tazawa-Shirakura'83: $t(r+1,r) \geqslant \frac{1}{r}$
- Chung-Lu'99: $t(r+1,r) \geqslant \frac{1}{r} + \frac{1}{r^2} + O(\frac{1}{r^3})$, odd r
- Lu-Zhao'09: $t(r+1,r) \geqslant \frac{1}{r} + \frac{1}{2r^3} + O(\frac{1}{r^4})$, $r \equiv 4 \mod 6$
- Double counting: $t(s,r) \geqslant \binom{s}{r}^{-1}$
- Spencer'72: improved for $s \gg r$
- De Caen'83: $t(s,r) \geqslant \binom{s-1}{r-1}^{-1}$

Upper bounds as $r \to \infty$

- $r \cdot t(r+1, r)$ is at most
  - Sidorenko'81: $O(\sqrt{r})$
  - Kim-Roush'83: $(2 + o(1)) \ln r$
  - Frankl-Rödl'85: $(1 + o(1)) \ln r$
  - Sidorenko'97: $(\frac{1}{2} + o(1)) \ln r$
- De Caen'94 (\$500): Does $r \cdot t(r+1, r) \to \infty$?
- P. ⩾24:

$$t(r+1, r) \leqslant \begin{cases} \frac{6.239}{r+1} & \text{all } r \\ \frac{4.911}{r+1} & \text{all } r \geqslant r_0 \end{cases}$$

- Frankl-Rödl'85: $\forall \rho \ \binom{r+\rho}{r} \cdot t(r+\rho, r) \lesssim \rho(\rho+4) \ln r$
- P. ⩾24: $\forall \rho \ \binom{r+\rho}{r} \cdot t(r+\rho, r) \leqslant \mu_\rho + o(1)$
  - $\forall \rho \geqslant \rho_0 \quad \mu_\rho \leqslant \rho \ln \rho + 3\rho \ln \ln \rho$

Connections to coding theory

- Alphabet Q of size q
- $C \subseteq Q^d$ is a ρ-insertion code if $\forall X \in Q^{d+\rho}$ $\exists Y \in C$ st X is Y plus ρ new symbols
- Each $Y \in Q^d$ gives $V^q(d,\rho) := \sum_{i=0}^{d} \binom{d+\rho}{i}(q-1)^i$ words $X \in Q^{d+\rho}$
- Lenz-Rashtchian-Siegel-Yaakobi'21: $\forall \rho$ $\forall d \gg \rho$

$$\min |C| \leqslant (\mathrm{e} + o(1))\rho \ln \rho \, \frac{q^{d+\rho}}{V^q(d,\rho)}$$

  - Cooper-Ellis-Kahng'02, Krivelevich-Sudakov-Vu'03
- Take $d \ll q$
  - $\{x_1, \ldots, x_d\} \mapsto$ all permutations of $(x_1, \ldots, x_d)$
  - $r = d$, $n = q$
  - Turán $(r+\rho, r)$-system $\mapsto$ symmetric ρ-insertion code
  - Idea: "symmetrise" construction of good codes
  - P.-Verbitsky-Zhukovskii' $\geqslant$24: new lower bounds on 1-insertion codes (for $d \ll q$)

High-level ideas for $(r+1, r)$-Turán systems

- Recursion
- Fix $k < r$
- Include $\{x_1 < \cdots < x_r\}$ depending on $\{x_1, \ldots, x_{k-1}\}$
  - Random choice for $\{x_1, \ldots, x_{k-1}\}$ (iid biased coins)
- For each "unhappy" k-set Y apply recursion on r-sets that start with Y

Formal proof

- Global constants μ, c, β
- Prove $T(n, r+1, r) \leqslant \frac{\mu}{r+1}\binom{n}{r}$ by induction on r and n
- $r \leqslant \mu - 1$: take $G_n^r = \binom{[n]}{r}$
- $r > \mu - 1$:
  - $k := \beta r$
  - S: $\frac{c}{k}$-random subset of $\binom{[n]}{k-1}$
  - $S^* := S \otimes K_*^{r-k+1} = \bigcup_{Y \in S}\{Y \cup Z : Z \in \binom{[\max Y+1, n]}{r-k+1}\}$
    - Extend each $Y \in S$ to the right to all possible r-sets
  - $T := \{Y \in \binom{[n]}{k} : \binom{Y}{k-1} \cap S = \emptyset\}$
  - $T^* := T \otimes G_*^{r-k} = \bigcup_{Y \in T}\{Y \cup Z : Z \in G_{n-\max Y}^{r-k}\}$
    - Extend each $Y \in T$ by Turán $(r-k+1, r-k)$-system
  - Claim: $G_n^r := S^* \cup T^*$ is a Turán $(r+1, r)$-system

Expected size of G_n^r

- S: $\frac{c}{k}$-random subset of $\binom{[n]}{k-1}$
- $S^* := S \otimes K_*^{r-k+1}$
- $\mathbb{E}|S^*| = \frac{c}{k}\binom{n}{r}$
- $T := \{Y \in \binom{[n]}{k} : \binom{Y}{k-1} \cap S = \emptyset\}$
- $T^* := T \otimes G_*^{r-k}$

$$
\begin{aligned}
\mathbb{E}|T^*| &= \sum_{y=k}^{n} \left(1 - \frac{c}{k}\right)^k \binom{y-1}{k-1} \cdot |G_{n-y}^{r-k}| \\
&\leqslant \sum_{y=k}^{n} \mathrm{e}^{-c} \binom{y-1}{k-1} \cdot \frac{\mu}{r-k+1} \binom{n-y}{r-k} \\
&= \frac{\mathrm{e}^{-c}\mu}{r-k+1} \binom{n}{r}
\end{aligned}
$$

Choosing appropriate constants

- Need (for all $r \geqslant \mu - 1$ with $k = \beta r$)

$$\frac{c}{k} + \frac{e^{-c}\mu}{r-k+1} \leqslant \frac{\mu}{r+1}$$

- E.g. $\beta := \frac{1}{2}$, $c := 1$, large μ works
- Large $r \geqslant r_0$:
  - Enough

$$\frac{c}{\beta} + \frac{e^{-c}}{1-\beta}\mu < \mu$$

  - $\beta = 0.715$, $c = 2.51 \;\Rightarrow\; \mu = 4.911$ suffices
  - Prove by induction on r and n that

$$T(n, r+1, r) \leqslant \left(\frac{\mu}{r+1} + \frac{D}{r\ln(r+3)}\right)\binom{r+\rho}{r}^{-1}\binom{n}{r}$$

Lower bounds on $t(r+\rho, r)$

- S: $c/\binom{k}{\rho}$-random subset of $\binom{n}{k-\rho}$
- $S^* := S \otimes K_*^{r-k+\rho}$
- $T := \{Y \in \binom{[n]}{k} : \binom{Y}{k-\rho} \cap S = \emptyset\}$
- $T^* := Y \otimes G_*^{n-k}$
- $G_n^r := S^* \cup T^*$
  - Turán $(r+\rho, r)$-system
- Need: $\frac{c}{\beta^\rho} + \frac{\mathrm{e}^{-c}}{(1-\beta)^\rho}\,\mu < \mu$
- $\mu := \frac{(c+1)^{\rho+1}}{c^\rho}$ where c is max root of $\mathrm{e}^c = (c+1)^{\rho+1}$
- $\rho \ln \rho < c \leqslant \rho \ln \rho + 2\rho \ln \ln \rho$ for $\rho \geqslant \rho_0$
- $\mu < \rho \ln \rho + 3\rho \ln \rho$ for all large ρ

Open problems

- Is $t(r+1, r) = (1 + o(1))\,\frac{1}{r}$?
- H_m^r: r-graph with $r+1$ vertices and m edges
  - $H_{r+1}^r = K_{r+1}^r$ so $\pi(H_{r+1}^r) = 1 - t(r+1, r)$
  - $\pi(H_i^r) = 0$ if $i = 1, 2$
  - $\pi(H_3^r) \leqslant 2/(r+1)$
  - Sidorenko'24: $\pi(H_3^r) \geqslant (1.721... + o(1))/r^2$
  - What is the correct power of r ?

Thank you!